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GRAHAM'S CONJECTURE ON $ZZ_n(C_{2k}) \times G$

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Abstract. The pebbling number of a graph G, $f(G)$, is the least m such that, however m pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves, each move taking two pebbles from one vertex and placing one on an adjacent vertex. We say that G satisfies the 2- *pebbling property* if for any distribution with more than $2f(G) - q$ pebbles, it is possible to move two pebbles to any specified vertex. Graham conjectured that for all graphs G and H, $f(G\times H) \leq f(G)f(H)$. Let $ZZ_n(C_{2k})$ be the zig zag chain graph of n copies of even cycles and let G be any graph with $2-$pebbling property. We prove that $f(ZZ_n(C_{2k})\times G) \leq f(ZZ_n(C_{2k}))f(G)$ for all $n \geq 2$.

Keywords: Graph Pebbling, Zig-zag chain graph.

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1. Introduction

Throughout this paper, unless stated otherwise, G will denote a simple connected graph. Suppose p pebbles are distributed onto the vertices of a graph G. A pebbling move consists of removing two pebbles from some vertex and adding one on an adjacent vertex. we say a pebble can be moved to a vertex v, the target vertex, if we can apply pebbling moves repeatedly so that in the resulting distribution we can move a pebble to the vertex v. To understand the pebbling concepts, we need the following definitions.

Definition 1.1. [1] [5] *The* pebbling number of a vertex v *in* G *is the smallest number* $f(G, v)$ *such that every placement of* $f(G, v)$ *pebbles, it is possible to move a pebble to* v *by a sequence* *of pebbling moves. Also, we define the* t-pebbling number of v *in* G *is the smallest number* $f_t(G, v)$ *such that from every placement of* $f_t(G, v)$ *pebbles, it is possible to move* t *pebbles to the vertex* v.

The pebbling number *of* G *and the* t-pebbling number *of* G *are the smallest numbers,* $f(G)$ *and* $f_t(G)$*, such that from any placement of* $f(G)$ *pebbles or* $f_t(G)$ *pebbles, respectively, it is possible to move one or* t *pebbles, respectively, to any target vertex by a sequence of pebbling moves. Thus,* $f(G)$ *and* $f_t(G)$ *are the maximum values of* $f(G, v)$ *and* $f_t(G, v)$ *over all vertices* v.

(1) For any vertex v of a graph G, $f(G, v) \geq n$ where $n = |V(G)|$
(2) The pebbling number of a graph G satisfies $f(G) \geq max\{2^{diam(G)}, |V(G)|\}$, where $diam(G)$ is the diameter of the graph G.

Definition 1.2. [1] [7] *Let* D *be a distribution of pebbles on* G*, let* q *be the number of vertices with at least one pebble. We say that* G *satisfies the* 2- pebbling property *if for any distribution with more than* $2f(G) - q$ *pebbles, it is possible to move two pebbles to any specified vertex.*

Further, we say that a graph G *has the* $2t$-pebbling property*, if for any distribution with more than* $2f_t(G) - q$ *pebbles, it is possible to move* $2t$ *pebbles to any specified vertex.*

The Cartesian product of G and H is denoted by $G \times H$. The following well-known conjecture is first appeared in [1].

Conjecture 1.3. [1] *For any connected graphs* G *and* H*,* $f(G \times H) \leq f(G)f(H)$.

Many articles (See, e.g.,[1],[2],[6] and [11]) have given evidence supporting Conjecture 1.3.In this paper we verified this conjecture is true for the product of zig-zag chain graph of n copies of even cycles, $ZZ_n(C_{2k})$ and the graph G with $2-$pebbling property. Further, Lourdusamy extended Conjecture 1.3 as follows.

Conjecture 1.4. [7] *For any connected graphs* G *and* H*,* $f_t(G \times H) \leq f_t(G)f(H)$

This paper is organized as follows. In Section 2, we give some preliminary pebbling results and definitions on zig-zag chain graph of n copies of even cycles. In section 3, we provide some lemmas that will be used in the proof of main results. In Section 4, we verify that Graham's Conjecture is true for the Cartesian product of zig-zag chain graph of n copies of even cycles and the graph G with 2-pebbling property.

2. PRELIMINARIES

Definition 2.1. [10] *The* zig-zag chain graph of n copies of even cycles *denoted by* $ZZ_n(C_{2k})$*, is a graph which consists of zig-zag sequence of* n *even cycles,* C_{2k} *with* $k \geq 3$*. We have the following vertex set and edge set of* $ZZ_n(C_{2k})$ *for* n *even as follows.*

$$V(ZZ_n(C_{2k})) = \{a_i, b_i : 1 \leq i \leq n(k-1)\} \cup \{x, y\} \text{ and}$$

$$E(ZZ_n(C_{2k})) = \{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n(k-1)-1\} \cup \{xa_1, xb_1, ya_{n(k-1)}, yb_{n(k-1)}\} \cup$$

$$\{a_{(k+1)i-1}b_{(k+1)i-2}, a_{(k+1)j}b_{(k+1)j+1} : 1 \leq i \leq \tfrac{n}{2}, 1 \leq j \leq (\tfrac{n}{2}-1)\}$$

For n odd, we have the following vertex set and edge set.

$V(ZZ_n(C_{2k})) = \{a_i, b_i : 1 \leq i \leq n(k-1)\} \cup \{x, y\}$ *and*

$E(ZZ_n(C_{2k})) = \{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n(k-1)-1\} \cup \{xa_1, xb_1, ya_{n(k-1)}, yb_{n(k-1)}\} \cup$

$$\{a_{(k+1)i-1}b_{(k+1)i-2}, a_{(k+1)j}b_{(k+1)j+1} : 1 \leq i, j \leq \tfrac{n-1}{2}\}.$$

The reader can easily view that $ZZ_n(C_{2k})$ has n copies of C_{2k}, and label each cycle as $A_1, A_2, ...,$ and A_n in order. Here, we present some results that will be used in the proof of main results.

Theorem 2.2. [7] *Let P_n be the path with n vertices. Then*
$(1) f_t(P_n) = t2^{n-1}$ *and*
$(2) P_n$ *satisfies the $2t$- pebbling property.*

Theorem 2.3. [8] [9] *Let C_{2k} denote a simple cycle with $2k$ vertices, where $n \geq 3$. Then*
(1)

$$f_t(C_{2k}) = \begin{cases} t2^k, & n \text{ is even} \\ \frac{2^{k+2}-(-1)^{k+2}}{3} + (t-1)2^k, & n \text{ is odd.} \end{cases}$$

(2) *The graph C_{2k} satisfies the $2t-$pebbling property.*

Theorem 2.4. [10] *Let $ZZ_n(C_{2k})$ be the zig-zag chain graph of n copies of even cycles. Then we have $f_t(ZZ_n(C_{2k})) = t.2^{n(k-1)}$.*

Theorem 2.5. [7] *Let P_n be the path with n vertices and let G be the graph with $2t-$pebbling property. We have $f_t(P_n \times G) \leq f_t(P_n) f(G)$.*

Theorem 2.6. *Let C_{2k} be the cycle with $2k$ vertices and let G be the graph with $2t-$pebbling property. We have $f_t(C_{2k} \times G) \leq f_t(C_{2k}) f(G)$.*

3. Useful Lemmas

In this section, we provide some lemmas which will be used in main results.

Lemma 3.1. *Let $ZZ_2(C_{2k})$ be the zig-zag chain graph of two copies of even cycles and let G be the graph with $2-$pebbling property. Suppose at least $(2^{2k-1} - 2^k) f(G)$ pebbles distributed only on the vertices of $A_1 \times G$. Then we can move at least $(2^{k-1} - 1)$ pebbles to $\{a_k\} \times G$.*

Proof. Consider the graph $ZZ_2(C_{2k})$ with at least $(2^{2k-1} - 2^k) f(G)$ pebbles distributed only on the vertices of $A_1 \times G$. We have to move at least $(2^{k-1} - 1)$ pebbles to $\{a_k\} \times G$. Clearly, $A_1 \times G \cong C_{2k} \times G$ and recall that $f_t(C_{2k} \times G) \leq f_t(C_{2k}) f(G)$. Therefore we can move at least $(2^{k-1} - 1)$ pebbles to $\{a_k\} \times G$. □

Lemma 3.2. *Let $ZZ_3(C_{2k})$ be the zig-zag chain graph of three copies of even cycles and let G be the graph with $2-$pebbling property. Suppose at least $(2^{3(k-1)+1} - 2^k)f(G)$ pebbles distributed only on the vertices of $(A_1 \cup A_2) \times G$. Then we can move at least $(2^{k-1} - 1)$ pebbles to $\{a_{2k-2}\} \times G$.*

Proof. Consider the graph $ZZ_3(C_{2k})$ with at least $(2^{3k-2} - 2^k)f(G)$ pebbles distributed only on the vertices of $(A_1 \cup A_2) \times G$. We have to move at least $(2^{k-1} - 1)$ pebbles to $\{a_{2k-2}\} \times G$. Suppose at least $(2^{2k-1} - 2^k)f(G)$ pebbles distributed on the vertices of $A_2 \times G$. Then by Lemma 3.1, we can move at least $(2^{k-1} - 1)$ pebbles to $\{a_{2k-2}\} \times G$. Therefore assume that $p(A_2 \times G) < (2^{2k-1} - 2^k)f(G)$. Then the number of pebbles retained on $A_1 \times G$ is at least $(2^{3k-2} - 2^{2k-1})f(G)$. Then we claim the following:

$$Claim(1) : p(A_1 \times G) \geq 2^k[2^{k-2}(2^{k-1} - 1)]f(G)$$

We have, $(2^{3k-2} - 2^{2k-1})f(G) - 2^k[2^{k-2}(2^{k-1} - 1)]f(G)$

$$= (2^{3k-2} - 2^{2k-1} - 2^{3k-3} + 2^{2k-2})f(G)$$

$$= (2^{3k-3} - 2^{2k-2})f(G)$$

$$> 0, \text{ since } k \geq 3.$$

Hence we can move at least $2^{k-2}(2^{k-1} - 1)$ pebbles to $\{a_k\} \times G$. Now, we have subgraph $A : \{a_k, a_{k+1}, ..., a_{2k-2}\}f(G) \cong P_{k-1} \times G$. Then by Theorem 2.2, we can move at least $(2^{k-1} - 1)$ pebbles to $\{a_{2k-2}\} \times G$. □

Lemma 3.3. *Let $ZZ_n(C_{2k})$ be the zig-zag chain graph of n copies of even cycles and let G be the graph with $2-$pebbling property. Suppose at least $(2^{n(k-1)+1})f(G)$ pebbles distributed only on the vertices of $\{A_1 \cup ... \cup A_{n-1}\} \times G$. Then we can move at least $(2^{k-1} - 1)$ pebbles to $\{a_{(n-1)(k-1)}\} \times G$.*

Proof. We prove this lemma by induction. For $n = 2$ and $n = 3$, the results follow from Lemma 3.1 and Lemma 3.2. Assume that the result is true for all $n^{'} < n$. Consider the graph $ZZ_n(C_{2k})$ with at least $(2^{n(k-1)+1})f(G)$ pebbles distributed only on the vertices of $\{A_1 \cup ... \cup A_{n-1}\} \times G$. We have to move at least $(2^{k-1} - 1)$ pebbles to $\{a_{(n-1)(k-1)}\} \times G$. Suppose at least $(2^{(n-1)(k-1)+1} - 2^k)f(G)$ pebbles distributed on the vertices of $(A_2 \cup ... \cup A_{n-1})f(G)$. Then by induction, we can move at least $(2^{k-1} - 1)$ pebbles to $\{a_{2k-2}\} \times G$. Therefore assume that $p((A_2 \cup ... \cup A_{n-1}) \times G) < (2^{(n-1)(k-1)+1} - 2^k)f(G)$. Then the number of pebbles retained on $A_1 \times G$ is at least $(2^{n(k-1)+1} - 2^{(n-1)(k-1)+1})f(G)$. We claim the following:

$$Claim(2) : p(A_1 \times G) \geq 2^k[2^{n(k-1)-2k+1}(2^{k-1} - 1)]f(G)$$

We have,

$$(2^{n(k-1)+1} - 2^{(n-1)(k-1)+1})f(G) - 2^k[2^{n(k-1)-2k+1(2^{k-1}-1)}]f(G)$$

$$= (2^{n(k-1)+1} - 2^{(n-1)(k-1)+1} - 2^{n(k-1)-k+1(2^{k-1}-1)})f(G)$$

$$= (2^{n(k-1)+1} - 2^{(n-1)(k-1)+1} - 2^{n(k-1)} + 2^{n(k-1)-k+1})f(G)$$

$$= (2^{n(k-1)} - 2^{n(k-1)-k})f(G)$$

$$> 0$$

Hence we can move at least $2^{n(k-1)-2k+1}(2^{k-1}-1)$ pebbles to $\{a_k\} \times G$. Now, we have subgraph $B : \{a_k, ..., a_{(n-1)(k-1)}\} \times G \cong P_{n(k-1)-2k+2} \times G$. Then by Theorem 2.2, we can move at least $(2^{k-1}-1)$ pebbles to $\{a_{(n-1)(k-1)}\} \times G$. □

4. Main results:

In this section, we verify that Graham's conjecture is true for the product of zig-zag chain graph of n copies of even cycles and a graph G satisfies the $2-$pebbling property.

Theorem 4.1. *Let $ZZ_2(C_{2k})$ be the zig-zag chain graph of n copies of even cycles and let G be the graph with $2-$pebbling property. Then*

$$f(ZZ_2(C_{2k}) \times G) \leq f(ZZ_2(C_{2k}))f(G).$$

Proof. Consider the graph $ZZ_2(C_{2k}) \times G$ with at least $2^{2k-1}f(G)$ pebbles distributed on its vertices. Let $(m, n) = v \in ZZ_2(C_{2k}) \times G$ be out target vertex. Here, $m \in ZZ_2(C_{2k})$ and $n \in G$. Let p_m denote the number of pebbles placed on the vertices of $\{m\} \times G$ and let q_m denote the number of occupied vertices in $\{m\} \times G$. Without loss of generality, assume that $v \in A_2 \times G$. We consider the following cases:

Case 1. Let $v \in (V(A_2) - \{y, b_{2(k-1)}\} \times G)$

Fix $v = (a_i, z), k \leq i \leq 2(k-1)$. Clearly, $A_2 \times G \cong C_{2k} \times G$. Suppose $p(A_2 \times G) \geq 2^k f(G)$.Then by Theorem2.3, we can reach the target. So assume that $p(A_2 \times G) < 2^k f(G)$. Then the number of pebbles retained on $A_1 \times G$ is at least $(2^{2k-1} - 2^k)f(G)$. By Lemma 3.1, we can move at least $(2^{k-1}-1)$ pebbles to (a_k, z) and by Theorem 2.2, we can move one pebble to the target vertex.

Case 2. Let $v \in \{y, b_{2(k-1)}\} \times G$

Without loss of generality, assume that $v \in \{y\} \times G$. Fix $v = (y, z)$. Now we have two subgraphs $I = \{a_k, a_{k+1}, ..., y\} \times G$ and $J = \{b_k, b_{k+1}, ..., y\} \times G$ which are isomorphic to $P_{k-1} \times G$. Suppose $p(I) \geq 2^{k-2}f(G)$ or $p(J) \geq 2^{k-2}f(G)$. Then we can reach the target. Otherwise, assume that $p(I) < 2^{k-2}f(G)$ and $p(J) < 2^{k-2}f(G)$. Without loss of generality, assume that all the pebbles are distributed only on the vertices of $A_1 \times G$. Then by Lemma 3.1, we can move at least $(2^{k-1}-1)$ pebbles to the vertex (a_k, z) by using exactly $(2^{2k-1}-2^k)f(G)$ pebbles. But the number of pebbles retained on $A_1 \times G$ is at least $2^k f(G)$. Therefore, we can move an additional pebble to the vertex (a_k, z). Now by using the subgraph I, we can move a pebble to the vertex (y, z). □

Theorem 4.2. *Let $ZZ_3(C_{2k})$ be the zig-zag chain graph of three copies of even cycles and let G be the graph with $2-$pebbling property. Then*

$$f(ZZ_3(C_{2k}) \times G) \leq f(ZZ_3(C_{2k}))f(G).$$

Proof. Consider the graph $ZZ_3(C_{2k}) \times G$ with at least $2^{3k-2}f(G)$ pebbles on the vertices. Let $v = (m, n) \in ZZ_3(C_{2k}) \times G$ be our target vertex. Here, $m \in ZZ_3(C_{2k})$ and $n \in G$. Let p_m denote the number of pebbles in $\{m\} \times G$ and let q_m denote the number of occupied vertices in $\{m\} \times G$. Without loss of generality, assume that $v \in A_t \times G, 1 \leq t \leq 3$. We consider the following cases:

Case 1. Let $v \in A_2 \times G$.

Suppose $p((A_2 \cup A_3) \times G) \geq 2^{2k-1}f(G)$.Then the number of pebbles retained on $A_1 \times G$ is at least $2^{2k-1}f(G)$. Therefore $p((A_1 \cup A_2) \times G) \geq 2^{2k-1}f(G)$. Again by Theorem 4.1, we can reach the target.

Case 2. Let $v \in A_1 \times G$ or $v \in A_3 \times G$.

Without loss of generality, let us take $v \in A_3 \times G$ and $p(A_3 \times G) < 2^k f(G)$. Then the number of pebbles distributed on the vertices of $(A_1 \cup A_2) \times G$ is at least $(2^{3k-2} - 2^k)f(G)$. We consider the following subcases:

Subcase 2(a). Let $v \in (A_3 - \{y, a_{3(k-1)}\}) \times G$

Without loss of generality, we assume that $v = (a_{3(k-1)}, z)$. Since, we have at least $(2^{3k-2} - 2^k)f(G)$ pebbles on the vertices of $(A_1 \cup A_2) \times G$. By Lemma 3.2, we can move at least $(2^{k-1} - 1)$ pebbles to the vertex $(a_{2(k-1)}, z)$. Then we can put one pebble to the target vertex $v = (a_{3(k-1)}, z)$.

Subcase 2(b). Let $v \in \{y, a_{3(k-1)}\} \times G$.

Without loss of generality, assume that $v \in \{y\} \times G$. Now we have two subgraphs $K : \{a_k, ..., y\} \times G$ and $L : \{b_k, ..., y\} \times G$ which are isomorphic to $P_{2(k-1)} \times G$. Suppose $p(K) \geq 2^{2(k-2)}f(G)$ or $p(L) \geq 2^{2(k-2)}f(G)$. Then we can reach the target. Otherwise, assume that $p(K) < 2^{2(k-2)}f(G)$ or $p(L) < 2^{2(k-2)}f(G)$.Without loss of generality assume that all the pebbles are distributed only on the vertices of $A_1 \times G$. Then by Lemma 3.2, we can move at least $(2^{k-1}-1)$ pebbles to the vertex $(a_{2(k-1)}, z)$ by using exactly $(2^{3k-2}-2^k)f(G)$ pebbles. But the number of pebbles retained on $A_1 \times G$ is at least $2^k f(G)$. Therefore we can move an additional pebble to the vertex $(a_{2(k-1)}, z)$. Now, by using the subgraph K we can move a pebble to the vertex (y, z).

□

Theorem 4.3. *Let $ZZ_n(C_{2k})$ be the zig-zag chain graph of n copies of even cycles and let G be the graph with $2-$pebbling property. Then*

$$f(ZZ_n(C_{2k}) \times G) \leq f(ZZ_n(C_{2k}))f(G).$$

Proof. We prove this theorem by induction on n. For $n=2$ and $n=3$, the result follows from Theorem 4.1 and Theorem 4.2. Assume that the result is true for all $n^{'} < n$. Consider the graph $ZZ_n(C_{2k})$ with at least $(2^{n(k-1)+1})f(G)$ pebbles on its vertices. Let $v \in A_t \times G, 1 \leq t \leq n$. We consider the following cases:

Case 1. Let $v \in A_t \times G, 1 < t < n$.

The graph $ZZ_n(C_{2k}) \times G$ can be partitioned into two subgraphs say, S_1 and S_2, where $S_1 \cong ZZ_p(C_{2k}) \times G$ and $S_2 \cong ZZ_s(C_{2k}) \times G$. Here, $n = s + p - 1$. Clearly, $S_1 \cap S_2 \cong A_t \times G$. Suppose $p(S_1) \geq 2^{p(k-1)+1}f(G)$.Then we are done. Therefore assume that $p(S_1) < 2^{p(k-1)+1}f(G)$. Then the number of pebbles retained on S_2 is at least $2^{s(k-1)+1}f(G)$ which implies $p(S_2) \geq 2^{s(k-1)+1}f(G)$. Then by induction we can reach the target vertex.

Case 2. Let $v \in A_1 \times G or A_n \times G$.

Without loss of generality, assume that $v \in A_n \times G$ and $p(A_n \times G) < 2^k f(G)$. Then the number of pebbles retained on $(A_1 \cup ... \cup A_{n-1}) \times G$ is at least $(2^{n(k-1)+1} - 2^k)f(G)$. We consider the following subcases:

Subcase 2(a). Let $v \in \{V(A_n) - \{y, b_{n(k-1)}\}\} \times G$.
Let us take $v = (a_{n(k-1)}, z)$. Since $p(A_1 \cup ... \cup A_{n-1}) \geq (2^{n(k-1)+1} - 2^k)f(G)$. Then by Lemma 3.3, we can move at least $(2^{k-1}-1)$ pebbles to $\{a_{(n-1)(k-1)}\} \times G$. Then we can reach the target.

Subcase 2(b). Let $v \in \{y, b_{n(k-1)}\} \times G$.
Without loss of generality, assume that $v = (y, z)$. We have two subgraphs say, $X : \{a_k, ..., a_{n(k-1)}\} \times G$ and $Y : \{b_k, ..., b_{n(k-1)}\} \times G$. Suppose $p(X) \geq 2^{n(k-1)-k+1}$ and $p(Y) \geq 2^{n(k-1)-k+1}$. Then we can move a pebble to the target vertex. Therefore assume that $p(X) < 2^{n(k-1)-k+1}$ and $p(Y) < 2^{n(k-1)-k+1}$. Without loss of generality, assume that all the pebbles are distributed only on the vertices of $A_1 \times G$. Then by Lemma 3.3, we can move at least $(2^{(n-1)k-n+1} - 1)$ pebbles to (a_k, z) by using exactly $2^k(2^{(n-1)k-n+1} - 1)f(G)$ pebbles. Now, we have at least $2^k f(G)$ pebbles retained on $A_1 \times G$. By using Theorem 2.6 we can move additional pebble to

the vertex (a_k, z). Then by Theorem 2.2, we can move one pebble to the vertex (y, z) through the subgraph X. □

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